# Magnetic Effects and Misc. Topics

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# Contents

# Chapter 1

# Electronic Properties in a Magnetic Field

## 1.1 The Boltzmann Equation

We define a **distribution function** $n(\vec{k}, \vec{r}, t)$ for the carriers such that $n(\vec{k}, \vec{r}, t)d\vec{r}d\vec{k}$ represents the number of carriers at the time $t$ in a volume element $d\vec{r}d\vec{k}$ of phase space, i.e. the number of carriers who share a neighbourhood around $\vec{r}$ and also have momentums within a very short range of $\hbar\vec{k}$. Note that we are dealing with vector quantities and so there are three components for both $\vec{r}$ and $\vec{k}$. If the total number of carriers in the system (all possible positions and all possible momentums) is $N$, then obviously:

$$\int\int n(\vec{k}, \vec{r}, t)d\vec{r}d\vec{k} = N \tag{1.1}$$

### 1.1.1 The Case of No Collisions

We choose a carrier at random. Around the representative point (in phase-space) of this carrier, we consider a region. The boundary of this region is demarkated by some other phase space points occupied by other carriers. Within this region, many other carriers' representative points are present. In the absense of collisions, the only change that can occur to the positions and velocities of the carriers are due to the applied fields or potentials which give rise to position and occasionally velocity-dependent forces. Now let us suppose that one of the representative points moves to the boundary and shares the position of one of the boundary points (this means that the carrier changes position and velocity so that it arrives at the same position and has the same velocity (*vector velocity - magnitude and direction*) as another carrier which has such a position and momentum as to place it on the boundary of the region). At this point, the forces and hence the accelerations that this carrier experiences are the same as that of the carrier on the boundary. Thus from this instant of time onwards, both the carriers will move together. What

this means is that a carrier within the region cannot go out of the region or another carrier from outside cannot come in. As long as the region is defined in terms of the carriers forming the boundary, this boundary cannot be crossed. So the number of carriers within this region is constant. Liouville's theorem states that the volume of this kind of phase space region does not change due to the application of forces derived from potentials. Thus, if we follow the motion of the representative point of one carrier from the time $t$ to the time $t + \delta t$ and consider the region around it defined as above, the region can change shape, but the volume will not change. Again the number of representative points within this region will also not change. Thus we can write:

$$n(\vec{k} + \delta\vec{k}, \vec{r} + \delta\vec{r}, t + \delta t)d(\vec{r} + \delta\vec{r})d(\vec{k} + \delta\vec{k}) = n(\vec{k}, \vec{r}, t)d\vec{k}d\vec{r}$$

From this, remembering that Liouville's theorem states that the volume of the phase space region does not change $[d(\vec{k} + \delta\vec{k}) = d(\vec{k})$ etc.] we get:

$$n(\vec{k} + \delta\vec{k}, \vec{r} + \delta\vec{r}, t + \delta t) - n(\vec{k}, \vec{r}, t) = 0$$

After making a Taylor expansion of the first, term, we get, for the **total** change in $n$ after time $\delta t$:

$$\frac{dn}{dt} = \left(\frac{\partial}{\partial t} + \frac{d\vec{r}}{dt} \cdot \vec{\nabla}_{\vec{r}} + \frac{d\vec{k}}{dt} \cdot \vec{\nabla}_{\vec{k}}\right) n(\vec{k}, \vec{r}, t) = 0$$

where, $\vec{\nabla}_{\vec{r}} \equiv \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$, the usual gradient operator and $\vec{\nabla}_{\vec{k}} \equiv \frac{\partial}{\partial k_x}\hat{i} + \frac{\partial}{\partial k_y}\hat{j} + \frac{\partial}{\partial k_z}\hat{k}$, a corresponding operator with respect to the $\vec{k}$ vector (note that in the phase-space description position and momentum are on an equal footing). Now $\vec{v} = \frac{d\vec{r}}{dt}$ and the force on the carrier, which is the rate of change of momentum is given by $\vec{F} = \hbar\frac{d\vec{k}}{dt}$. Incorporating these in the above equation we get:

$$\frac{\partial n}{\partial t} + \vec{v} \cdot \vec{\nabla}_{\vec{r}} n + \frac{\vec{F}}{\hbar} \cdot \vec{\nabla}_{\vec{k}} n = 0 \tag{1.2}$$

This is the *Collisonless Boltzmann Equation* and it regulates the flow of representative points of carriers in phase-space. Note that the 'motion' of the phase-space point system, the 'dynamics' of which is described by this equation, is not the same as the actual motion of the carriers. Here we have mapped even the velocity of the carriers as a kind of 'coordinate' in a kind of higher dimensional space.

### 1.1.2 The Effect of Collisions

We consider only energy-conserving (elastic) collisions. Due to these collisions, some of the representative points will be scattered out of the region and some other points will come in from the outside. The collision term $\left(\frac{\partial n}{\partial t}\right)_{coll}$ causes a

change in $n$ even if we flow along with a particle. The generalized Boltzmann equation is then:

$$\left(\frac{\partial n}{\partial t}\right) + \underbrace{\vec{v}\cdot\vec{\nabla}_{\vec{r}}n}_{I} + \underbrace{\frac{\vec{F}}{\hbar}\cdot\vec{\nabla}_{\vec{k}}n}_{II} = \left(\frac{\partial n}{\partial t}\right)_{coll} \tag{1.3}$$

The term $I$ is due to the non-uniformity of the carrier distribution in space. It is called the **diffusion** or **drift** term and generally arises due to thermal gradients. The term $II$ is called the **force term** and is due to the applied external fields.

**To calculate $\left(\frac{\partial n}{\partial t}\right)_{coll}$:**

Since we are considering Fermions where two particles cannot occupy the same state, a particle can be scattered from a state $\vec{k}$ to another state $\vec{k}'$ only if $\vec{k}'$ is empty. Then the expectation value for transition from state $\vec{k}'$ to $\vec{k}$ (my present state) is proportional to the following quantities:

- $n_{\vec{k}'} \longrightarrow$ the number of states available for scattering from
- $\left(1 - n_{\vec{k}}\right) \longrightarrow$ the number of states available to scatter into $\cdots$ the 1 comes because being Fermions only one particle is allowed per state. This means that this number can be one or zero - either the state is empty and available or filled and unavailable.
- $Q(\vec{k},\vec{k}') = Q(\vec{k}',\vec{k}) \longrightarrow$ the microscopic probability per unit time that an electron *known to be accupying the state* $\vec{k}$ be scattered into the state $\vec{k}'$ *known to be unoccupied*. The equality implies the microscopic reversibility of the transition from $\vec{k}$ to $\vec{k}'$.

To get the net increase $\left(\frac{\partial n}{\partial t}\right)_{coll}$ , we must take the difference of the number scattered in and the number scattered out and integrate over all the *other* $\vec{k}'$ s (our $\vec{k}$ is fixed). Thus:

$$\left(\frac{\partial n}{\partial t}\right)_{coll} = \int Q(\vec{k},\vec{k}')\{n_{\vec{k}'}(1-n_{\vec{k}}) - n_{\vec{k}}(1-n_{\vec{k}'})\}d\vec{k}' = I(n) \tag{1.4}$$

Here $n_{\vec{k}} \equiv n(\vec{k},\vec{r},t)$. $I(n)$ is called the **collision integral**.

When **thermal equilibrium** is established, collisions *alone* do not change the total density of representative points at any particular point in the phase diagram (thermal motion being random) and $I(n)$ vanishes. From (1.4) we then get:

$$\int Q(\vec{k},\vec{k}')\{n^0_{\vec{k}'}(1-n^0_{\vec{k}})\}d\vec{k}' = \int Q(\vec{k},\vec{k}')\{n^0_{\vec{k}}(1-n^0_{\vec{k}'})\}d\vec{k}' \tag{1.5}$$

where $n^0_{\vec{k}}$ is the symbol used to denote the thermal equilibrium state. Putting $P(\vec{k},\vec{k'}) = Q(\vec{k},\vec{k'})n^0_{\vec{k}}(1-n^0_{\vec{k'}})$, we get, from the above equation:

$$P(\vec{k},\vec{k'}) = P(\vec{k'},\vec{k}) \tag{1.6}$$

This is called the **principle of detailed balance** for a system in thermal equilibrium.

**Assumption**

The **steady state** distribution $n(\vec{k},\vec{r},t)$ in a field under isothermal conditions does not depart very far from the equilibrium distribution $n^0_{\vec{k}}$:

$$\delta n(\vec{k},\vec{r},t) = n(\vec{k},\vec{r},t) - n^0_{\vec{k}} \tag{1.7}$$

Applying equation (1.3) on (1.7), we get

$$\begin{aligned} n(\vec{k},\vec{r},t) &= \delta n(\vec{k},\vec{r},t) + n^0_{\vec{k}} \\ \Rightarrow \frac{\partial}{\partial t}(\delta n_{\vec{k}} + n^0_{\vec{k}}) &= -\vec{v}\cdot\vec{\nabla}_{\vec{r}}(\delta n_{\vec{k}} + n^0_{\vec{k}}) - \frac{\vec{F}}{\hbar}\cdot\vec{\nabla}_{\vec{k}}(\delta n_{\vec{k}} + n^0_{\vec{k}}) \\ &\quad + I(\delta n_{\vec{k}} + n^0_{\vec{k}}) \end{aligned}$$

At equilibrium (steady state) $\frac{\partial}{\partial t}n^0_{\vec{k}} = 0$. Then we get:

$$\begin{aligned} \frac{\partial}{\partial t}\delta n_{\vec{k}} &= -\vec{v}\cdot\vec{\nabla}_{\vec{r}}\delta n_{\vec{k}} - \vec{v}\cdot\vec{\nabla}_{\vec{r}}n^0_{\vec{k}} - \frac{\vec{F}}{\hbar}\cdot\vec{\nabla}_{\vec{k}}\delta n_{\vec{k}} \\ &\quad -\frac{\vec{F}}{\hbar}\cdot\vec{\nabla}_{\vec{k}}n^0_{\vec{k}} + I(\delta n) \end{aligned}$$

where $I(\delta n) = \int Q(\vec{k},\vec{k'})(\delta n_{\vec{k}} - \delta n_{\vec{k'}})d\vec{k'}$. Since the diffusion term vanishes at equilibrium, $\vec{v}\cdot\vec{\nabla}_{\vec{r}}n^0_{\vec{k}} = 0$. In the force term, $n^0_{\vec{k}}$ being much larger than $\delta n_{\vec{k}}$, only that contribution remains. Again, in the collision term, detailed balance causes the term in $n^0_{\vec{k}}$ to vanish and we ultimately get the **linearized Boltzmann equation**:

$$\frac{\partial}{\partial t}\delta n_{\vec{k}} = -\vec{v}\cdot\vec{\nabla}_{\vec{r}}\delta n_{\vec{k}} - \frac{\vec{F}}{\hbar}\cdot\vec{\nabla}_{\vec{k}}n^0_{\vec{k}} + I(\delta n) \tag{1.8}$$

### 1.1.3 Relaxation Time Approximation

We assume that if $\delta n$ is caused by the application of an external field to the equilibrium distribution and at $t = 0$ we switch off the external field, $\delta n$ decays exponentially with a **relaxation time** $\tau$ such that:

$$\delta n(\vec{k},\vec{r},t) = \delta n(\vec{k},\vec{r},0)e^{-\frac{t}{\tau}} \tag{1.9}$$

Now, under isothermal conditions, (no diffusion term) and zero field equation (1.8) becomes:

$$\begin{aligned} \frac{\partial}{\partial t}\delta n_{\vec{k}} &= I(\delta n) \\ \Rightarrow I(\delta n) &= -\frac{\delta n_{\vec{k}}}{\tau} \end{aligned} \tag{1.10}$$

Now the relaxation is a function of the collisions undergone by the system and can be reasonably assumed to exist even when there are applied fields. This is because the relaxation mechanism (collisions) does not know whether there is a field or not. The collisions are thermal in nature - they are caused by the random thermal motion of the carriers. Any drift velocity caused by applied fields is very small compared to the random velocity ($\approx cm/s$ as compared to $\approx 100\, m/s$) and so the collision process is affected only marginally by the external influences. The collision mechanism tries to bring the system back to the equilibrium configuration with the same dynamics as it did when the field was switched off suddenly. So there are two influences here. The field tries to change the configuration of the system from the equilibrium distribution and collisions try to restore it, with a 'restoring force' which is a measure of the relaxation time (smaller the relaxation time greater the restoring 'force'). In fact, if this were not so, the system would accelerate continuously under the influence of the applied field and we would not get the phenomenon of resistivity. (Note that this is the statistical mechanics analog of the kinetic theory trick that we built into the Drude model of conductivity when we asserted that the effect of collisions is to make a carrier forget all its past and go back to the equilibrium distribution). Thus we can use the same relaxation time form of the collision integral even in the Boltzmann equation with field. Now $\frac{\partial}{\partial \vec{k}} n^0_{\vec{k}} = \frac{\partial}{\partial E} n^0_{\vec{k}} \times \frac{\partial E}{\partial \vec{k}}$. Again, since $E = \frac{\hbar^2}{2m} k_x^2 + k_y^2 + k_z^2$,

$$\frac{\partial E}{\partial \vec{k}} = \frac{\hbar^2}{2m}(2k_x\hat{i} + 2k_y\hat{j} + 2k_z\hat{k}) = \frac{\hbar^2}{m}\vec{k}$$

Then the force term in the Boltzmann equation becomes:

$$\begin{aligned} -\tfrac{\vec{F}}{\hbar} \cdot \tfrac{\partial}{\partial \vec{k}} n^0_{\vec{k}} &= -\tfrac{\vec{F}}{\hbar} \cdot \tfrac{\partial}{\partial E} n^0_{\vec{k}} \tfrac{\hbar^2}{m}\vec{k} \\ &= -\vec{F} \cdot \tfrac{\hbar\vec{k}}{m} \tfrac{\partial}{\partial E} n^0_{\vec{k}} \\ &= -\vec{F} \cdot \vec{v} \tfrac{\partial}{\partial E} n^0_{\vec{k}} \end{aligned} \tag{1.11}$$

Again $\delta n_{\vec{k}} = -I(\delta n)\tau$ from (1.10). Thus under *constant field, isothermal condition* and *steady state* we get, from the linearized Boltzmann equation (1.8) and (1.10):

$$\begin{aligned} I(\delta n) - \tfrac{\vec{F}}{\hbar} \cdot \vec{\nabla}_{\vec{k}} n^0_{\vec{k}} &= 0 \\ \Rightarrow\ -\tfrac{\delta n_{\vec{k}}}{\tau} - \vec{F} \cdot \vec{v} \tfrac{\partial}{\partial E} n^0_{\vec{k}} &= 0 \\ \Rightarrow\ \delta n_{\vec{k}} &= -\vec{F} \cdot \vec{v}\tau \tfrac{\partial n^0_{\vec{k}}}{\partial E} \end{aligned} \tag{1.12}$$

where $n^0_{\vec{k}} \longrightarrow$ **Fermi function** for the electron.

## 1.2 The Boltzmann Equation in a Magnetic Field

In the linearized Boltzmann equation (1.8) we use (1.10) and keep all other terms to get:

$$\frac{\partial \delta n_{\vec{k}}}{\partial t} = -\vec{v} \cdot \frac{\partial \delta n_{\vec{k}}}{\partial \vec{r}} - \vec{F} \cdot \frac{\partial n^0_{\vec{k}}}{\hbar \partial \vec{k}} - \frac{\delta n_{\vec{k}}}{\tau} \tag{1.13}$$

Here $\vec{F}$ is the Lorentz force:

$$\vec{F} = -\mid e \mid \vec{E} - \frac{\mid e \mid}{c} \vec{v} \times \vec{B} \tag{1.14}$$

When this is substituted directly into equation (1.12) we get a term in $(\vec{v} \times \vec{B}) \cdot \vec{v}$ which vanishes identically. So the zeroth order term will not suffice. This is because no work is done by the magnetic force on a charge. The speed i.e. kinetic energy of the electron does not change. So the magnetic field has to be included exactly. We take the case of an isothermal system in the steady state and the equation (1.13) reduces to:

$$\vec{F} \cdot \frac{\partial n^0_{\vec{k}}}{\hbar \partial \vec{k}} + \frac{\delta n_{\vec{k}}}{\tau} = 0$$

In place of $\vec{F}$ we have to use the Lorentz force. Using the symbol $\vec{\varepsilon}$ for the electric field we write the terms in expanded form and make the following approximations as we did earlier in the zero-magnetic-field case:

1. For the electric field, all the carriers interact (because the electric field can impart energy) and so the equilibrium distribution $n_{\vec{k_0}}$ is used as the change in the distribution is small (in line 3 first term).

2. For the magnetic term, only those carriers which move are relevant and hence only the departure term $\delta n_{\vec{k}}$ is kept (in line 4 second term).

3. The derivative with respect to $\vec{k}$ replaced with a derivative w.r.t. $E$ as in (1.11) above.

Thus we get (the sign is included in $e$):

$$\begin{aligned}
\vec{F} \cdot \tfrac{\partial n^0_{\vec{k}}}{\hbar \partial \vec{k}} + \tfrac{\delta n_{\vec{k}}}{\tau} &= 0 \\
\Rightarrow \left[ e\vec{\varepsilon} - \tfrac{e}{c} \vec{v} \times \vec{B} \right] \cdot \tfrac{1}{\hbar} \tfrac{\partial n_{\vec{k}}}{\partial \vec{k}} + \tfrac{\delta n_{\vec{k}}}{\tau} &= 0 \\
\Rightarrow \tfrac{e}{\hbar} \vec{\varepsilon} \cdot \tfrac{\partial n^0_{\vec{k}}}{\partial \vec{k}} - \tfrac{e}{\hbar c} (\vec{v} \times \vec{B}) \cdot \tfrac{\partial n_{\vec{k}}}{\partial \vec{k}} + \tfrac{\delta n_{\vec{k}}}{\tau} &= 0 \\
\Rightarrow e\vec{v} \cdot \vec{\varepsilon} \tfrac{\partial n^0_{\vec{k}}}{\partial E} + \left( \tfrac{e}{\hbar c} \right) \vec{v} \times \vec{B} \cdot \tfrac{\partial \delta n_{\vec{k}}}{\partial \vec{k}} + \tfrac{\delta n_{\vec{k}}}{\tau} &= 0
\end{aligned} \tag{1.15}$$

Now we try to find a solution to this equation which is similar in form to (1.12):

$$\delta n_{\vec{k}} = -\tau e \vec{v} \cdot \vec{D} \frac{\partial n^0_{\vec{k}}}{\partial E} \tag{1.16}$$

Our task is to find a vector $\vec{D}$ such that the above equation is really a solution of (1.15). We try:

$$\vec{\varepsilon} = \vec{D} + (\omega_c \tau)\vec{b} \times \vec{D} \tag{1.17}$$

where $\vec{b} = \frac{\vec{B}}{|\vec{B}|}$, a unit vector along the direction of $\vec{B}$ and $\omega_c = \frac{eB}{mc}$ is the cyclotron frequency. If we substitute $\vec{\varepsilon}$ from (1.17) and $\delta n_{\vec{k}}$ from (1.16) into (1.15), and remembering $\vec{v} = \frac{\hbar \vec{k}}{m}$, we indeed find that the whole thing becomes zero:

$$\begin{aligned}
& e\vec{v} \cdot [\vec{D} + (\omega_c\tau)\vec{b} \times \vec{D}] \tfrac{\partial n^0_{\vec{k}}}{\partial E} + \tfrac{e}{\hbar c}\vec{v} \times \vec{B} \cdot \tfrac{\partial \delta n_{\vec{k}}}{\partial \vec{k}} + \tfrac{\delta n_{\vec{k}}}{\tau} \\
= \;& e\vec{v} \cdot \left[\vec{D} + (\omega_c\tau)\tfrac{\vec{B}\times\vec{D}}{|B|}\right] \tfrac{\partial n^0_{\vec{k}}}{\partial E} + \tfrac{e}{\hbar c}\vec{v} \times \vec{B} \tfrac{\partial}{\partial \vec{k}} \underbrace{\left[-\tau e \vec{v} \cdot \vec{D} \frac{\partial n^0_{\vec{k}}}{\partial E}\right]}_{\text{From defn of } \vec{\varepsilon} \text{ w.r.t. } \vec{D}.} - e\vec{v} \cdot \vec{D} \frac{\partial n^0_{\vec{k}}}{\partial E} \\
= \;& e\vec{v} \cdot \vec{D} \overbrace{\frac{\partial n^0_{\vec{k}}}{\partial E}}^{\vec{I}} + \tfrac{ev\cdot(\vec{B}\times\vec{D})}{|B|}(\omega_c\tau)\tfrac{\partial n^0_{\vec{k}}}{\partial E} + \tfrac{e}{\hbar c}(\vec{v} \times \vec{B})\tfrac{\partial}{\partial \vec{k}}\left[-\tau e \vec{v} \cdot \vec{D} \tfrac{\partial n^0_{\vec{k}}}{\partial E}\right] - e\vec{v} \cdot \vec{D} \overbrace{\frac{\partial n^0_{\vec{k}}}{\partial E}}^{\vec{I}} \\
= \;& \tfrac{e\vec{v}\cdot(\vec{B}\times\vec{D})}{|B|}\tfrac{\partial n^0_{\vec{k}}}{\partial E} \underbrace{\frac{e|\vec{B}|}{mc}}_{=\omega_c}\tau + \tfrac{e}{\hbar c}(\vec{v} \times \vec{B})\tfrac{\partial}{\partial \vec{k}}\left[-\tau e \underbrace{\frac{\hbar \vec{k}}{m}}_{=\vec{v}} \cdot \vec{D} \tfrac{\partial n^0_{\vec{k}}}{\partial E}\right] \\
= \;& \tfrac{e^2\tau\vec{v}\cdot(\vec{B}\times\vec{D})}{mc}\tfrac{\partial n^0_{\vec{k}}}{\partial E} - \tfrac{e^2\tau}{mc}(\vec{v} \times \vec{B}) \cdot \vec{D} \tfrac{\partial n^0_{\vec{k}}}{\partial E} \\
= \;& \tfrac{e^2\tau}{mc}(\vec{v} \times \vec{B}) \cdot \vec{D} \tfrac{\partial n^0_{\vec{k}}}{\partial E} - \tfrac{e^2\tau}{mc}(\vec{v} \times \vec{B}) \cdot \vec{D} \tfrac{\partial n^0_{\vec{k}}}{\partial E} \; \{since\ \vec{v} \cdot (\vec{B} \times \vec{D}) = \vec{D} \cdot (\vec{v} \times \vec{B})\} \\
= \;& 0
\end{aligned}$$

Thus, $\vec{D}$, as defined in (1.17), will really lead to the solution (1.16) of the linearized Boltzmann equation with magnetic field. However, to find $\vec{D}$, the equation (1.17) will have to be inverted to give $\vec{D}$ in the form:

$$\vec{D} = a_1\vec{\varepsilon} + a_2\vec{\varepsilon} \times \vec{B} + a_3(\vec{B} \cdot \vec{\varepsilon})\vec{B} \tag{1.18}$$

where the $a_i s$ have to be chosen accordingly. Putting (1.18) into (1.17) we get:

$$\begin{aligned}
\vec{\varepsilon} &= a_1\vec{\varepsilon} + a_2(\vec{\varepsilon}\times\vec{B}) + a_3(\vec{B}\cdot\vec{\varepsilon})\vec{B} \\
&\quad +(\omega_c\tau)\vec{b}\times[a_1\vec{\varepsilon} + a_2(\vec{\varepsilon}\times\vec{B}) + a_3(\vec{B}\cdot\vec{\varepsilon})\vec{B}] \\
&= a_1\vec{\varepsilon} + a_2(\vec{\varepsilon}\times\vec{B}) + a_3(\vec{B}\cdot\vec{\varepsilon})\vec{B} \\
&\quad -a_1\omega_c\tau(\vec{\varepsilon}\times\vec{b}) + a_2\omega_c\tau\{\vec{b}\times(\vec{\varepsilon}\times\vec{B})\} + a_3\omega_c\tau(\vec{B}\cdot\vec{\varepsilon})\underbrace{\vec{b}\times\vec{B}}_{=0} \\
&= a_1\vec{\varepsilon} + a_2(\vec{\varepsilon}\times\vec{B}) + a_3(\vec{B}\cdot\vec{\varepsilon})\vec{B} - a_1\omega_c\tau(\vec{\varepsilon}\times\vec{b}) \\
&\quad +a_2\omega_c\tau(\vec{b}\cdot\vec{B})\vec{\varepsilon} - a_2\omega_c\tau(\vec{b}\cdot\vec{\varepsilon})B \ (using\ \vec{A}\times(\vec{B}\times\vec{C})\ identity) \\
&= [a_1 + a_2\omega_c\tau(\vec{b}\cdot\vec{B})]\vec{\varepsilon} + [a_2|\vec{B}| - a_1\omega_c\tau](\vec{\varepsilon}\times\vec{b}) \\
&\quad +\{a_3|\vec{B}|(\vec{b}\cdot\vec{\varepsilon}) - a_2\omega_c\tau(\vec{b}\cdot\vec{\varepsilon})\}\vec{B}
\end{aligned} \tag{1.19}$$

Now if we choose:

$$|\vec{B}|a_3(\vec{b}\cdot\vec{\varepsilon}) - a_2\omega_c\tau(\vec{b}\cdot\vec{\varepsilon}) = 0$$

then:

$$\begin{aligned}
a_3|\vec{B}| &= a_2\omega_c\tau \\
\Rightarrow a_3|\vec{B}| &= a_2\tfrac{e|\vec{B}|}{mc}\tau \ \ (since\ \omega_c = \tfrac{e|\vec{B}|}{mc}) \\
\Rightarrow a_3 &= a_2\tfrac{e\tau}{mc}
\end{aligned} \tag{1.20}$$

Then (1.19) becomes:

$$\vec{\varepsilon} = [a_1 + a_2\omega_c\tau|\vec{B}|]\vec{\varepsilon} + [a_2|\vec{B}| - a_1\omega_c\tau](\vec{\varepsilon}\times\vec{b})$$

Since $\vec{\varepsilon}$ and $(\vec{\varepsilon}\times\vec{b})$ are orthogonal vectors, we get:

$$a_1 + a_2\omega_c\tau|\vec{B}| = 1 \tag{1.21}$$

and:

$$a_2|\vec{B}| = a_1\omega_c\tau \tag{1.22}$$

From (1.22):

$$\begin{aligned}
a_2|\vec{B}| &= a_1\tfrac{e|\vec{B}|}{mc}\tau \\
\Rightarrow a_2 &= \tfrac{e\tau}{mc}a_1
\end{aligned} \tag{1.23}$$

Substituting this into (1.21) we get:

$$\begin{aligned}
a_1\left[1 + \tfrac{e\tau}{mc}\omega_c\tau\tfrac{\omega_c mc}{e}\right] &= 1 \quad (since\ |\vec{B}| = \tfrac{\omega_c mc}{e}) \\
\Rightarrow a_1\left[1 + (\omega_c\tau)^2\right] &= 1 \\
\Rightarrow a_1 &= \tfrac{1}{1+(\omega_c\tau)^2} \\
and\ a_2 &= \tfrac{e\tau}{mc}\tfrac{1}{1+(\omega_c\tau)^2} \\
and\ a_3 &= \tfrac{e\tau}{mc}a_2 = \left(\tfrac{e\tau}{mc}\right)^2\tfrac{1}{1+(\omega_c\tau)^2}
\end{aligned} \tag{1.24}$$

We use these values of the $a$'s in eqn. (1.16) to get:

$$\begin{aligned}
\delta n_{\vec{k}} &= -\tau e\vec{v}\cdot\vec{D}\frac{\partial n^0_{\vec{k}}}{\partial E} \\
&= -e\tau\vec{v}\cdot[a_1\vec{\varepsilon}+a_2(\vec{\varepsilon}\times\vec{B})+a_3(\vec{B}\cdot\vec{\varepsilon})\vec{B}]\frac{\partial n^0_{\vec{k}}}{\partial E} \\
&= -e\tau\vec{v}\cdot\frac{\left[\vec{\varepsilon}+\frac{e\tau}{mc}|\vec{B}|(\vec{\varepsilon}\times\vec{b})+\left(\frac{e\tau}{mc}\right)^2|\vec{B}|^2(\vec{b}\cdot\vec{\varepsilon})\vec{b}\right]}{1+(\omega_c\tau)^2}\frac{\partial n^0_{\vec{k}}}{\partial E} \\
&= -e\tau\vec{v}\cdot\frac{\left[\vec{\varepsilon}+\frac{e\tau}{mc}\frac{\omega_c mc}{e}(\vec{\varepsilon}\times\vec{b})+\left(\frac{e\tau}{mc}\right)^2\frac{\omega_c^2(mc)^2}{e^2}(\vec{b}\cdot\vec{\varepsilon})\vec{b}\right]}{1+(\omega_c\tau)^2}\frac{\partial n^0_{\vec{k}}}{\partial E} \\
\Rightarrow \delta n_{\vec{k}} &= -e\tau\vec{v}\cdot\frac{\vec{\varepsilon}+(\omega_c\tau)(\vec{\varepsilon}\times\vec{b})+(\omega_c\tau)^2(\vec{b}\cdot\vec{\varepsilon})\vec{b}}{1+(\omega_c\tau)^2}\frac{\partial n^0_{\vec{k}}}{\partial E}
\end{aligned} \tag{1.25}$$

### 1.2.1 The Current density

For a free particle in a volume $v = L^3$ the allowed values of $k_x$, $k_y$ and $k_z$ are spaced $\frac{2\pi}{L}$ apart (using periodic boundary condition $e^{ik_x x} = e^{ik_x(x+L)}$, etc.). A region in $\vec{k}$-space of volume $\Omega$ will contain $\frac{\Omega}{(2\pi/L)^3} = \frac{\Omega V}{8\pi^3}$ allowed values of $\vec{k}$. Thus the number of allowed values per unit volume of $\vec{k}$-space is $\frac{V}{(2\pi)^3}$ where $V$ is the real space volume, i.e. sample size. For any smooth function $F(\vec{k})$,

$$\sum_{\vec{k}} F(\vec{k}) = \frac{V}{(2\pi)^3}\sum_{\vec{k}} F(\vec{k})\Delta\vec{k}$$

where we have replaced the summation over specific $\vec{k}$s with a summation over the $\vec{k}$-space volume remembering to multiply by the proper weight, i.e. the number of such states per unit $\vec{k}$-space volume. In the limit of infinite volume the summation gets replaced by an integral:

$$lim_{V\to\infty}\frac{1}{V}\sum_{\vec{k}} F(\vec{k}) = \int \frac{1}{(2\pi)^3}F(\vec{k})d\vec{k} \tag{1.26}$$

Now, the current density is due to the *excess carriers* flowing in a particular direction (the carriers within $n^0_{\vec{k}}$ flowing in a particular direction are compensated by an equal number flowing in the opposite direction - that is the meaning of equilibrium!). Only the distortion in the distribution function caused by the application of the fields will thus cause transport. Thus, the current density:

$$\vec{J}(\vec{r},t) = \sum_{\vec{k}} \delta n_{\vec{k}}(\vec{k},\vec{r},t)e\vec{v}$$

There are two electrons (spins) per $\vec{k}$ state. Thus:

$$\vec{J}(\vec{r},\vec{t}) = \frac{2e}{(2\pi)^3}\int \vec{v}\delta n(\vec{k},\vec{r},t)d\vec{k} \quad (F(\vec{k}) = e\vec{v}) \tag{1.27}$$

Now we use (1.16) to substitute $\delta n_{\vec{k}}$:

$$\vec{J}(\vec{r},t) = \frac{2e^2}{(2\pi)^3}\int \tau\vec{v}\left(-\frac{\partial n^0_{\vec{k}}}{\partial E}\right)\vec{v}\cdot\vec{D}d\vec{k} \tag{1.28}$$

We write this in the form:

$$\vec{J} = \alpha\vec{\varepsilon} + \beta\vec{\varepsilon}\times\vec{B} + \gamma(\vec{B}\cdot\vec{\varepsilon})\vec{B} \tag{1.29}$$

By inserting $\delta_n$ from (1.25) into (1.27), a comparison with (1.29) yields:

$$\begin{array}{rclcl} \alpha &=& \frac{2e^2}{3}\int \frac{\tau v^2(-\partial n^0_{\vec{k}}/\partial E)}{1+(\omega_c\tau)^2}d\vec{k} &=& \frac{\sigma_0}{1+(\omega_c\tau)^2} \\ \beta &=& \frac{2e^2}{3}\int \frac{\tau(\omega_c\tau/B)v^2(-\partial n^0_{\vec{k}}/\partial E)}{1+(\omega_c\tau)^2}d\vec{k} &=& \frac{\sigma_0(\omega c\tau/B)}{1+(\omega_c\tau)^2} \\ \gamma &=& \frac{2e^2}{3}\int \frac{\tau(\omega_c\tau/B)^2v^2(-\partial n^0_{\vec{k}}/\partial E)}{1+(\omega_c\tau)^2}d\vec{k} &=& \frac{\sigma_0(\omega_c\tau/B)^2}{1+(\omega_c\tau)^2} \end{array} \tag{1.30}$$

Here $\sigma_0$ is the conductivity for $\vec{B}=0$, because in that case equation (1.29) becomes $\vec{J}=\alpha\vec{\varepsilon}$ and $\omega_c\tau$ i.e $\omega_c = \frac{eB}{mc}$ becomes zero. Taken together these equations give the simple well-known form:

$$\vec{J} = \sigma_0\vec{D} \tag{1.31}$$

with $\vec{D}$ replacing $\vec{\varepsilon}$. The most useful form of the result from the experimental point of view is the relation between $\vec{\varepsilon}$ and $\vec{J}$, got by substituting $\vec{D}$ from equation (1.31) into equation (1.17), remembering that $\rho_0 = \sigma_0^{-1}$:

$$\begin{array}{rcl} \vec{\varepsilon} &=& \vec{D} + \frac{\omega_c\tau}{B}\vec{B}\times\vec{D} \\ &=& \rho_0\vec{J} + \frac{\omega_c\tau\rho_0}{B}\vec{B}\times\vec{J} \end{array} \tag{1.32}$$

Thus we see that for a given current and magnetic field, there are two components in the electric field $\varepsilon_\parallel$ and $\varepsilon_\perp$. If the magnetic field $\vec{B}$ is applied parallel to $\vec{J}$, as in the case of a long straight wire, $\vec{B}\times\vec{J}$ will be zero and we will have:

$$\varepsilon_\parallel = \rho_0 J \tag{1.33}$$

and:

$$\varepsilon_\perp = RBJ \tag{1.34}$$

where $R = \frac{\omega_c\tau\rho_0}{B}$ is the *Hall coefficient.*

These results show that in a single-band isotropic model there is *no magnetoresistance.* The Hall effect, however, is seen. The results are the same as found from classical considerations.

## 1.3 Hall Effect and Magnetoresistance in a Two Band Model

In real solids, the constant energy surfaces are of two types, one being the electron surface and the other being the hole surface since both these carriers exist at finite temperatures. We assume that the total current is made up of two current components $\vec{J}_1$ and $\vec{J}_2$. Now since both these types of carriers see the same electric and magnetic fields, equation (1.32) gives:

$$\begin{aligned} \vec{\varepsilon} &= \sigma_1^{-1}\vec{J}_1 + R_1\vec{B}\times\vec{J} \\ \vec{\varepsilon} &= \sigma_2^{-1}\vec{J}_2 + R_2\vec{B}\times\vec{J} \end{aligned} \tag{1.35}$$

where the symbols have the usual meanings. Again, $\vec{J} = \vec{J}_1 + \vec{J}_2 = \sigma_1\vec{D}_1 + \sigma_2\vec{D}_2$. We take $\vec{\varepsilon}$ perpendicular to $\vec{B}$ and then in equation (1.29) the third term vanishes and from equations (1.30) (to give $\alpha$ and $\beta$) we get:

$$\begin{aligned} \vec{J} &= \left[\frac{\sigma_1}{1+(\omega_{1c}\tau_1)^2} + \frac{\sigma_2}{1+(\omega_{2c}\tau_2)^2}\right]\vec{\varepsilon} \\ &+ \left[\frac{R_1\sigma_1^2}{1+(\omega_{1c}\tau_1)^2} + \frac{R_2\sigma_2^2}{1+(\omega_{2c}\tau_2)^2}\right]\vec{\varepsilon}\times\vec{B} \end{aligned} \tag{1.36}$$

We now assume for simplicity that there are $N_e$ electrons and $N_h$ holes having the same mass $m$ and the same relaxation time $\tau$ but equal and opposite (sign) cyclotron frequencies (the sign of the charges being opposite). Putting:

$$\sigma_0 = \frac{(N_e + N_h)e^2\tau}{m} \tag{1.37}$$

and remembering that $R$ has a $\sigma$ in the denominator, the bracketed part of the second term in (1.36) becomes, using $\sigma_1 = \frac{N_e e^2\tau}{m}$ and $\sigma_2 = \frac{N_h e^2\tau}{m}$ and $R_1 = -R_2$ :

$$\frac{\frac{\omega_c\tau}{B}(N_e - N_h)\frac{e^2\tau}{m}}{1+(\omega_c\tau)^2} = \frac{\frac{\omega_c\tau}{B}(N_e - N_h)(N_e + N_h)\frac{e^2\tau}{m}}{[1+(\omega_c\tau)^2](N_e + N_h)} = \frac{(N_e - N_h)\sigma_0\frac{\omega_c\tau}{B}}{(N_e + N_h)[1+(\omega_c\tau)^2]}$$

and using this equation (1.36) becomes:

$$\vec{J} = \frac{\sigma_0}{1+(\omega_c\tau)^2}\vec{\varepsilon} + \left(\frac{N_e - N_h}{N_e + N_h}\right)\frac{\sigma_0(\omega_c\tau/B)}{1+(\omega_c\tau)^2}\vec{\varepsilon}\times\vec{B} \tag{1.38}$$

### 1.3.1 The Hall Coefficient in the two-band model

To get the Hall coefficient the equation (1.38) must be inverted to express $\vec{\varepsilon}$ in terms of $\vec{J}$ and $\vec{J}\times\vec{B}$. On doing this the Hall coefficient becomes:

$$R = \frac{(_e - N_h)[1+(\omega_c\tau)^2]}{ec[(N_e + N_h)^2 + (N_e - N_h)^2(\omega_c\tau)^2]} \tag{1.39}$$

Thus for $N_e = N_h$, $R = 0$. For $N_e \neq N_h$, this becomes, in low fields such that $\omega_c\tau \ll 1$ for closed orbits:

$$R = \frac{(N_e - N_h)}{(N_e + N_h)^2 ec} \equiv \frac{R_1\sigma_1^2 + R_2\sigma_2^2}{(\sigma_1 + \sigma_2)^2} \tag{1.40}$$

### 1.3.2 The Magnetoresistance in the two-band model

The magnetoresistance is a tensor, but only two components of this tensor are of special interest. These are the *longitudinal magnetoresistance* corresponding to $\vec{J}$ being parallel to $\vec{B}$ and the *transverse magnetoresistance* corresponding to $\vec{J}$ being perpendicular to $\vec{B}$. Since the two-band model is isotropic and the contribution to $\vec{J}$ from $\vec{B}$ comes in the form of $\vec{\varepsilon} \times \vec{B}$ (1.38), The longitudinal component vanishes. Since we have taken $\vec{\varepsilon}$ perpendicular to $\vec{B}$, to compute the transverse component, we need to evaluate the component of $\vec{\varepsilon}$ parallel to $\vec{J}$ and compare the result with $\vec{J}$.

$$\begin{aligned} \rho &= \frac{\vec{J}\cdot\vec{\varepsilon}}{J^2} \\ &= \frac{1+(\omega_c\tau)^2}{\sigma_0}\left[1 + \left(\frac{N_e - N_h}{N_e + N_h}\right)^2 (\omega_c\tau)^2\right]^{-1} \end{aligned} \tag{1.41}$$

The fractional increase over the zero-field resistivity $\rho_0 (\equiv 1/\sigma_0)$ defines the magnetoresistance:

$$\frac{\Delta\rho}{\rho_0} \equiv \frac{\rho - \rho_0}{\rho_0} = \frac{4(\omega_c\tau)^2 N_e N_h}{(N_e + N_h)^2 + (N_e - N_h)^2(\omega_c\tau)^2} \tag{1.42}$$

This is a positive number. As the magnetic field, i.e. $\omega_c$ is increased, the magnetoresistance $\Delta\rho/\rho_0$ increases quadratically at first, and then, relatively slowly. The saturation value of the magnetoresistance occurs when (for closed orbits):

$$\frac{\Delta\rho}{\rho_0} = \frac{4N_eN_h}{N_e - N_h}$$

for $N_e \neq N_h$. For $N_e = N_h$, the magnetoresistance is proportional to $B^2$.

#### Kohler's rule

The scattering machanism and magnetic fields come into the picture through the term $\omega_c\tau$. At low fields, the scattering is important and $\Delta\rho$ is proportional to $\omega_c\tau$. Since $\omega_c$ is proportional to $B$ and $\tau$ is proportional to $\frac{1}{\rho}$:

$$\frac{\Delta\rho}{\rho_0} = F\left(\frac{B}{\rho_0}\right)$$

where $F$ is a function (some function) dependent on the nature of the metal only. Thus the quantity of interest is $\frac{B}{\rho_0}$, they always occur in this form. We can plot the magnetoresistance in one universal diagram if the samples are from the same metal. This is called *Kohler's rule*.