NOTES ON GROUP THEORY

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Chapter 1

Abstract Group Theory

1.1 Some definitions and fundamental concepts

1.1.1 Internal operation

Operations may be defined between two elements of a set i.e. $ab = c$ (a operates on b). If the operation is defined for **all** elements a and **all** elements b in a set and if c is always in the same set, then the operation is internal (to that set).

Note: all elements a $\Rightarrow$ all elements may operate
all elements b $\Rightarrow$ all elements may be operated on

1.1.2 Associativity

A set S is said to be associative under a given operation if, $(ab)c = a(bc)$

1.1.3 Commutativity

$ab = ba$

1.1.4 Regular element

if $ax = ay$ for all $x, y \in S \Rightarrow x = y$ then a is a regular element of S. e.g. Of the set of integers, 0 is regular under addition $\rightarrow 0 + a = 0 + b \Rightarrow a = b$ but not under multiplication $0 \times a = 0 \times b \,! \Rightarrow a = b$

1.1.5 Right and left unit elements

$ae = a$, $ea = a$

1.1.6 Right and left inverse (or symmetric element)

$aa^{-1} = e$, $a^{-1}a = e$

Theorems

1. If a and b are *regular elements* of S under an *associative* operation, ab is also regular.
   R.T.P. If $(ab)x = (ab)y$, $x = y$ (from the definition of regularity)
   From associativity, $(ab)x = (ab)y \Rightarrow a(bx) = a(by)$
   But a is regular $\Rightarrow bx = by$
   b is also regular $\Rightarrow x = y$
   Note: We have to use the regularity of both a and b!

2. If both unit elements exist, then they are equal.
   Proof: If $f \equiv$ left unit element and $e \equiv$ right unit element, apply f on $e \rightarrow fe = e$. Again apply e on f (from the right) $fe = f$! Therefore, $e = f$.

3. If associativity exists, then the right and left inverses are equal (if they exist).
   $a_R^{-1} = fa_R^{-1} = (a_L^{-1}a)a_R^{-1} = \underbrace{a_L^{-1}(aa_R^{-1})}_{associativity} = a_L^{-1}e = a_L^{-1}$

4. If associativity exits and if a_L^{-1} exist, then a is regular (corollary to 3)
   Proof: Let $ax = ay$ $x, y \in S$. Apply the left inverse of a to both sides: $a^{-1}(ax) = a^{-1}(ay) \Rightarrow \underbrace{(a^{-1}a)x = (a^{-1}a)y}_{associativity} \Rightarrow ex = ey \Rightarrow x = y$. Thus a is regular.

5. If associativity exists and b^{-1} and a^{-1} exist, $(ab)^{-1} = b^{-1}a^{-1}$
   Proof: $(b^{-1}a^{-1})(ab) = b^{-1}(a^{-1}a)b = b^{-1}eb = b^{-1}b = e \Rightarrow b^{-1}a^{-1} = (ab)^{-1}$. Similarly by multiplying from the right by $(b^{-1}a^{-1})$.

1.1.7 Group

- Closure - All products after operating one element on another must belong to the group

- Associativity - As seen above, this if very fundamental - exceptions are quite rare - vector triple products do not obey associativity

- Existence of both right and left unit elements (they are the same!)
- Existence of inverse - due to associativity right-left is immaterial

The number of elements of a group is called its **order**. A commutative group is also called **Abelian**. Two groups with the same multiplication table are **isomorphic** if the relation is one-one. If more than one element of a group is associated with the same element of another group and then if the multiplication table holds then the two groups are **homomorphic** and the mapping is many-one. This is a simpler requirement because since more than one element is associated with one element of the other group, all that is required is that the result of multiplying two elements should give rise to any element which is associated with the corresponding product element of the smaller group. Thus the trivial example consists of the homomorphism between **any** group and the group containing only the identity element. Since all the elements are associated with the identity, any product will also map onto the identity!

1.2 Rearrangement theorem and cyclic groups

1.2.1 Rearrangement theorem

Each row or column of any group multiplication table contains **each** element once and once only. The elements are only **rearranged** if they are multiplied by the same element. This means that each row or column of the group multiplication table contains the whole group (note that a row or column means that the element that labels the row or column is multiplied by all the other group elements), but the orders are different.
Proof: Consider the row labeled by the element a. Let b_1 and b_2 belong to this group such that $ab_1 = c$ and $ab_2 = c$ (i.e. two c's appear in the same row). Multiply from the left by a^{-1}, $b_1 = a^{-1}c$, $b_2 = a^{-1}c$. By closure $a^{-1}c$ belongs to the group. Therefore $b_1 = b_2$, i.e. different elements cannot give same product with one element. Thus c can appear only once in a row. Now there are as many rows or elements as the order of the group (because all the elements appear as labels on both rows and columns of the multiplication table). Now if each has to be different then the whole group has to be exhausted! As a corollary, every row or column of the group multiplication table has to be different. Otherwise the same element would appear more than once in some column or row. Note that for an Abelian group the multiplication table is symmetric about the diagonal.

1.2.2 Cyclic groups

If we multiply a group element by itself, because of closure, the result must also belong to the group. If the result is the same as the element, then the element must be the identity element. If it is different, then successive multiplication (with the same element $\equiv x$) will throw up different elements (since no two elements in the row or column labeled by x may be identical - by the rearrangement theorem). Since the row or column must contain all the elements, the identity must also occur at some time. After that we would be back with x and the sequence will be repeated. The size of this sequence may be smaller than the whole group. At the most, the size of the sequence may be the same, it cannot be larger (since the identity element must occur within each row or column of the multiplication table).

If $x^n = E$, then the set $x, x^2, \cdots, x^{n-1}, x^n$ is called the **period** of x and n is the **order** of x. The period of x forms a group by itself. It need not be the same group (if the period is smaller). Accordingly it is called a **cyclic group** or **cyclic subgroup**. All cyclic groups have to be Abelian as $x \cdot x^2 = x \cdot x \cdot x = x^2 \cdot x$.

1.3 Examples of groups

1.3.1 Symmetry group of an equilateral triangle

We start with Tinkham's example, the symmetries of an equilateral triangle. The operations are shown in fig. 1.1

Here, E means no change and A is a reflection about A (dotted line) or rotation through $\pi\,(180^o)$ about A. B and C are the corresponding operations about the other dotted axes. D is a clockwise rotation through $\frac{2\pi}{3}\,(120^o)$ in the plane of the paper and F is a similar rotation $\frac{4\pi}{3}\,(240^o)$ clockwise or $\frac{-2\pi}{3}\,(120^o)$ anticlockwise. A note on **convention** - for successive operations, the axes are assumed fixed in space, i.e. they do not rotate or change position with the object. This means that there is no *real* numbering of the axes, in fact otherwise how could one talk of symmetrical transformations? However, these positions are not indistinguishable to the extent that they do not form different states. This means that the symmetry is with respect to some particular variable for which these positions are equivalent. However, there are other variables which determine that the states are really different. This is like states having same energy but different momenta. As far as energy is considered, all the states are degenerate, but momentum eigenvalues may be vastly different. If this was not the case then we would have got a smaller

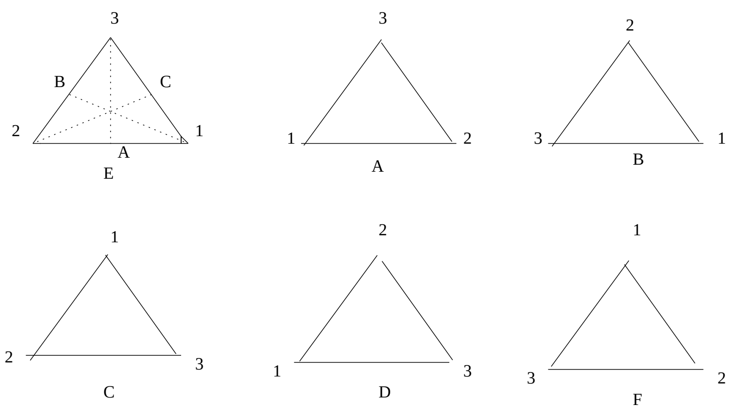

Figure 1.1: The symmetry operations on an equilateral triangle

number of operations, as in the case of indistinguishable particles. We give below the multiplication table for this group (row times column):

	E	**A**	**B**	**C**	**D**	**F**
E	E	A	B	C	D	F
A	A	E	D	F	B	C
B	B	F	E	D	C	A
C	C	D	F	E	A	B
D	D	C	A	B	F	E
F	F	B	C	A	E	D

A group isomorphic with the one given above is:

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, B = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, C = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix},$$

$$D = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, F = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

The identity element forms a cyclic subgroup of order unity. The period of $A = A$, $A^2 = E$. Thus order of A, B and C is two. Again, the period of $D = D$, $D^2 (= F)$, $D^3 (= DF = E)$. Thus D is of order 3 and D, E, F form a cyclic subgroup of our entire group of order 6. Note that this is just a manifestation of the fact that three successive rotations, maintaining the symmetry at each stage (i.e. the angle gets fixed at 120^o) bring the triangle back to itself whereas only two reflections do the same, but the successive reflections have to be about the same axis. The reflections are self-contained about each axis, the axes do not mix. The rotations, however, do not form such separate collective entities. All the rotations thus get lumped into one cyclic subgroup.

1.3.2 A list of common groups and standard symbols (from Falicov)

	Symbol	**Elements**	**Operation**	E	a^{-1}	**Properties**
	Z	Integers	Addition	0	$-a$	Abelian
	Z_n	Remainder (m/n)	Addition mod. n	0	$-a+n$	Cyclic
	Q	Rational numbers	Addition	0	$-a$	Abelian
	Q^*	Rational numbers $\neq 0$	Mult.	1	$1/a$	Abelian
	R	Real numbers	Addition	0	$-a$	Abelian
	R^*	Real numbers $\neq 0$	Mult.	1	$1/a$	Abelian
	C	Complex numbers	Addition	0	$-a$	Abelian
	C^*	Complex numbers $\neq 0$	Mult.	1	$\frac{1}{a} = \frac{a^*}{\|a\|^2}$	Abelian
	S	Complex no.s on unit circle	Mult.	1	$\frac{1}{a} = a^*$	Abelian
	S_n	n^{th} roots of unity	Mult.	1	$\frac{1}{a} = a^*$	Iso. to Z_n
Matrices:						
General Linear	$GL(n)$	$n \times n$ matrices	Mat. Mult.	δ_{ij}	a^{-1}	
Gen. lin. real	$GL(nR)$	$n \times n$ real matrices	Mat. Mult.	δ_{ij}	a^{-1}	
Special linear	$SL(n)$	$n \times n$ matrices, det.=1	Mat. Mult.	δ_{ij}	a^{-1}	
Unitary	$U(n)$	$n \times n$ unitary matrices	Mat. Mult.	δ_{ij}	$a^{\dagger}$	
Special unitary	$SU(n)$	$n \times n$, unitary, det.=1	Mat. Mult.	δ_{ij}	$a^{\dagger}$	
Orthogonal	$O(n)$	$n \times n$ real unitary	Mat. Mult.	δ_{ij}	$\tilde{a}$	

1.3.3 Examples of finite groups (from Falicov)

- Order 1.

	E
E	E

- Order 2. Cyclic, isomorphic to S_2 (1, -1) and Z_2 (0, 1)

	E	A
E	E	A
A	A	A

- Order 3. Cyclic, isomorphic to S_3, Z_3

	E	A	B
E	E	A	B
A	A	B	E
B	B	E	A

- Order 4. Cyclic, isomorphic to S_4, Z_4

	E	A	B	C
E	E	A	B	C
A	A	B	C	E
B	B	C	E	A
C	C	E	A	B

- Order 4. Non-cyclic since $X^2 = E$, Abelian

	E	A	B	C
E	E	A	B	C
A	A	E	C	B
B	B	C	E	A
C	C	B	A	E

- Order 5. Cyclic, isomorphic to S_5, Z_5
- Order 6. There is one cyclic group isomorphic to S_6, Z_6 and another non-Abelian group isomorphic with Tinkham's example (symmetries of triangle or permutations of a triplet). **This is the lowest order non-Abelian group**.

1.4 Subgroups

To verify whether a set is a subgroup of some other group, all the group properties need not be verified. The properties to check are:

1. All elements A in the subgroup S are also in the group G.

2. Whenever A, $B \in S$, $AB^{-1} \in S$

Proof: Put $B = A$. Therefore $AB^{-1} = AA^{-1}$ is in S. Thus the identity element is there.

Now put $A = E$. $AB^{-1} = EB^{-1} = B^{-1}$is then also in S. So inverse is also there

Now B^{-1} is in S. Then $A\left(B^{-1}\right)^{-1} = AB$ is also in S. So closure is shown.

Associativity is true for all elements of the original group.

So all the group properties are satisfied.

1.5 Cosets

A **complex** is a set of elements from a group G considered as a whole, i.e a complex is a subset (not necessary subgroup!). Now if A is a complex and a is an element of G, then the complex aA is the **left product** of a with A. The **right product** is defined similarly.

Let S be a subgroup of G. If $x \in G \wedge x \notin S$, then the complex xS is a left coset of S and Sx is the corresponding right coset (if $x \in S$ then xS would be S itself, due to the rearrangement theorem!).

1.5.1 Theorems

1. Cosets are not subgroups, they do not contain the identity element and Sx contains **no element in common** with the subgroup S.
   Proof: Let S_k belong to S. Also let $S_k x = S_l$, another element of S. Then $x = S_k^{-1} S_l$. But both $S_k \wedge S_l \in S$. So $S_k^{-1} \in S$ and also $S_k^{-1} S_l \in S$. Thus $x \in S$ and Sx is just S itself!

2. Two right or left cosets of $S \subset G$ are either identical or have no elements in common.
   Proof: Consider two cosets Sx and Sy. Let the common element be

$S_k x = S_l y$.

$$\begin{aligned}
S_k x &= S_l y \\
\Rightarrow S_k^{-1} S_k x &= S_k^{-1} S_l y \\
\Rightarrow x &= S_k^{-1} S_l y \\
\Rightarrow xy^{-1} &= S_k^{-1} S_l y y^{-1} \\
\Rightarrow xy^{-1} &= S_k^{-1} S_l \ \ (which\, belongs\, to\, S) \\
But\ Sxy^{-1} &= S\, (by\, rearrangement\, th.) \\
\Rightarrow Sxy^{-1}y &= Sy \\
\Rightarrow Sx &= Sy
\end{aligned}$$

So the two cosets are completely identical if just one common element exists! Thus, a finite group can be written as a **factored set**, $G = S + Sx + Sy + \cdots$ with the subgroup S and a finite number of *distinct* cosets. Note that since e is in S and *any* element x multiplied by e is $= x$, so x has either to belong to S or to *some* coset, the worst case being the coset formed by itself! Thus the factored set exhausts all the elements of G. No elements appear twice in the factored set (no element can be repeated in a coset because it can appear only once in a row or column of the multiplication table). So each coset contains n elements if n is the order of S. If m is the order of G, then $m = n \times l$ where l is an integer. Thus we get the following theorem:

3. *The order of a subgroup must be an integral divisor of the order of the entire group.*
   e.g. From Tinkham:- The subgroup A, E of order 2.

1.6 Class

1.6.1 Conjugate elements

Element b is conjugate to element a if, for a member x, $b = xax^{-1}$ or $a = x^{-1}bx$, which is the same thing.

1.6.1.1 Theorem

If b and c are both conjugate to a, they are conjugate to each other.
Proof: Let $b = xax^{-1}$, and $c = yay^{-1}$. Then $a = y^{-1}cy$. Replace a in the

first relation. Thus $b = xax^{-1} = xy^{-1}cyx^{-1} = (xy^{-1})cyx^{-1}$. But $yx^{-1}xy^{-1} = e \Rightarrow yx^{-1} = (xy^{-1})^{-1}$. Thus $b = (xy^{-1})c(xy^{-1})^{-1} = zcz^{-1}$, where $z \in G$.

All mutually conjugate elements form a **class**. To find the class containing a_i, form all products of the form:

$$\underbrace{ea_ie^{-1}}_{a_i},\ a_2a_ia_2^{-1},\ a_3a_ia_3^{-1},\ \cdots$$

Discard repeated elements. All elements of a group may be thus divided among the distinct classes. Each element must belong to a class because at least eae^{-1} is there! Again, and element a_i cannot belong to two classes because then, since all other elements of the two classes will have to be conjugate to a_i. Then by the above theorem they will all be conjugate to each other and hence belong to the same class. The identity element is in a class by itself as $aea^{-1} = aa^{-1} = e$. Since the elements of all classes have none in common, no other class can contain e and so only e is a class that is also a subgroup. In Abelian groups, each element is in a class by itself as:

$$\underbrace{xa}_{interchange}\quad x^{-1} = axx^{-1} = ae = a$$

If matrices are used to represent a group, then conjugation is equivalent to making a similarity transformation which leaves the trace invariant. So all elements in the same class will have the same trace. Physically, $b = x^{-1}ax$ means - operate by x, operate by a, in a way undo the operation by x. To allow such a course of action, b , the final equivalent operation and a , the original operation, should be of the same **kind**. The operation x, however, may be of a different kind, because whatever it does to fundamentally change the system will be undone by its inverse.

Tinkham's example: Consider the group of covering operations of the equilateral triangle we considered above. Consider the conjugation of A with D. Now $D^{-1}AD = C$. D rotates the triangle clockwise by 120^o so that vertex 2 instead of 3 lies on axis A; next the rotation by π about the axis A (or reflection in the axis A) interchanges 1 and 3; finally $D^{-1} = F$ rotates the triangle back by 120^o. This sequence has the same result as a single rotation of π about the C axis, which is an axis equivalent to A but rotated 120^o counterclockwise by the symmetry operator D^{-1}.

1.7 Invariant subgroup or normal divisor

If a subgroup S of G consists entirely of complete classes of G it is called an invariant subgroup or normal divisor. Here, if $a \in S$, all elements $x^{-1}ax$ (even if x belongs to G but does not belong to S) are also in S. It is unchanged by conjugation with any element of G, so it is *invariant*.

1.7.1 Theorem

Right and left cosets of an invariant subgroup are the same. (This may also be taken to be the definition of invariant subgroups).
Proof: For an invariant subgroup S, $x^{-1}Sx = S$ by the rearrangement theorem. Multiplying from the left by x, ($x \in G$, all x's considered) $Sx = xS$.

1.7.2 Factor group

Let S be an invariant subgroup. Now there are $l - 1$ distinct cosets (the subgroup itself is the other element). Let coset $\Re_i = Sk_i = k_iS\,(k_i \in G)$. Now

$$S\Re_i = S(Sk_i) = \underbrace{(SS)}_{S}k_i = Sk_i = \Re_i$$

Here $SS \equiv S$ because of the rearrangement theorem - we take each element only once in this complex. Thus S acts like an identity element. Together with S, this set of $l - 1$ complexes with $l = \frac{h}{g}$ forms a group, called the **factor group of G**. The group multiplication table of the factor group is homomorphic to that of G, because:

$$\Re_i\Re_j = (Sk_i)(Sk_j) = \underbrace{k_iS}_{left\,coset=right\,coset} Sk_j = k_i\underbrace{S}_{SS=S}k_j = S(k_ik_j) = (\Re_i\Re_j)$$

where $(\Re_i\Re_j)$ is the coset associated with k_ik_j.

Chapter 2

Representation Theory

2.1 Representations by matrices

If a set of matrices Γ can be found such that $\Gamma(A)\Gamma(B) = \Gamma(AB)$ where A, B are group elements and $\Gamma(A)$ is the matrix corresponding to A etc., then these matrices are said to form a representation of the group. If the representation is isomorphic, then the mapping is one-one and the representation is said to be **faithful** or **true**. The number of rows and columns is called the **dimension** of the representation. Square matrices are necessary because both the operations corresponding to AB and BA have to be defined. The determinants of the matrices also form a representation (non-matrix representation!) because the determinant of a product of matrices is the product of the determinants of the individual matrices. Since more than one of the matrices may have the same determinant, such a representation may be homomorphic. Similarity transformations leave matrix equations unchanged. If $\Gamma'(A) = S^{-1}\Gamma(A)S$, then:

$$\Gamma'(A)\Gamma'(B) = [S^{-1}\Gamma(A)\underbrace{S][S^{-1}}_{E}\Gamma(B)S] = S^{-1}\Gamma(A)\Gamma(B)S = S^{-1}\Gamma(AB)S = \Gamma'(AB)$$

Thus the transformed matrices also form a representation. Such representations, related by similarity transformations, are called **equivalent**.

2.2 Some properties of matrices and nomenclature

1. The *determinant* of the product of two matrices is equal to the product of the determinants.

2. Matrix multiplication is *associative*.

3. If $\mid \alpha \mid \neq 0$ a unique *inverse* exists for α.

4. All diagonal matrices *commute* and have a diagonal product.

5. The *trace* or *spur* of a matrix is the sum of its diagonal elements. The trace of a product is independent of the order of the factors:

$$Tr\,\alpha\beta = \sum_i (\alpha\beta)_{ii} = \sum_i \sum_j \alpha_{ij}\beta_{ji} = \sum_j \sum_i \beta_{ji}\alpha_{ij} = \sum_i (\beta\alpha)_{jj} = Tr\,\beta\alpha$$

6. A *similarity transformation* applied to a matrix α yields one of the form $\beta^{-1}\alpha\beta$. Such transformations preserve the trace as:

$$Tr\,\beta^{-1}(\alpha\beta) = Tr\,(\alpha\beta)\beta^{-1} = Tr\,\alpha$$

7. If all matrices in a matrix equation are subjected to the same similarity transformation the equation will hold for the transformed matrices also.

8. The *transpose* of a matrix α, denoted by $\tilde{\alpha}$, is obtained from α by reflecting across the principal diagonal, so that $\tilde{\alpha}_{ij} = \alpha_{ji}$. $\widetilde{(\alpha\beta)} = \tilde{\beta}\tilde{\alpha}$.

9. The *complex conjugate* of a matrix α, denoted by α^*, is obtained by taking the complex conjugate of all the elements. Here products stay in the same order as there is no transposition.

10. The *adjoint* of a matrix α, denoted by $\alpha^\dagger$, is the transpose of the complex conjugate. The order of the products change on taking the adjoint.

11. A *Hermitian* matrix is *self-adjoi*nt. Thus $H_{ij} = H^*_{ji}$.

12. If the inverse and adjoints are the same, the matrix is *unitary*. A unitary matrix with all real elements is called *real orthogonal*. In such cases $R^{-1} = R^\dagger = \tilde{R}$ and $(R^{-1})_{ij} = R_{ji}$. Transformations between orthogonal coordinate systems are carried out by such matrices.

13. The rows or columns of any unitary matrix form a set of n orthonormal vectors.

14. The product of two unitary matrices is unitary since:

$$(UV)^\dagger = V^\dagger U^\dagger = V^{-1}U^{-1} = (UV)^{-1}$$

15. The eigenvalues of Hermitian matrices are real. The eigenvalues of unitary matrices are of absolute value unity. If the matrix is really orthogonal, the eigenvalues appear in complex conjugate pairs. Both unitary and Hermitian matrices can always be diagonalised by similarity transformations using unitary matrices. This can be seen as follows: Consider the *secular equation* for the matrix α:

$$0 = \mid \alpha - \lambda E \mid = \begin{vmatrix} (\alpha_{11} - \lambda) & \alpha_{12} & \cdots \\ \alpha_{21} & (\alpha_{22} - \lambda) & \cdots \\ \cdots & \cdots & (\alpha_{nn} - \lambda) \end{vmatrix}$$

Now obtain another matrix by a similarity transformation $\beta = s^{-1}\alpha s$. Consider the secular equation of β.

$$\begin{aligned} 0 &= \mid \beta - \lambda E \mid = \mid s^{-1}\alpha s - \lambda E \mid = \mid s^{-1}\alpha s - \lambda s^{-1} E s \mid = \mid s^{-1}(\alpha - \lambda E) s \mid \\ &= \mid s^{-1} \mid\mid \alpha - \lambda E \mid\mid s \mid = \mid \alpha - \lambda E \mid\mid s^{-1} s \mid = \mid \alpha - \lambda E \mid \end{aligned}$$

Thus all the roots of the secular equation, which are the *eigenvalues* λ_k of the matrix, are invariant under a similarity transformation. For these values there are nontrivial solutions to the matrix equation:

$$\alpha v^k = \lambda_k v^k \tag{2.1}$$

The *eigenvectors* v^k are defined by this equation. They are unchanged in direction by operation with α. Thus they define the principal axes of the system. The eigenvalues λ_k give the degree of stretching of the eigenvector v^k produced by the operation α. For a Hermitian matrix, the eigenvectors are orthogonal. This can be seen as follows:- consider two instances of the eigenvalue equation:

$$Hv^i = \lambda_i v^i \, and \, Hv^k = \lambda_k v^k$$

Take the adjoint of the second equation and use the fact that for Hermitian matrices the adjoint is the same as the original matrix:

$$v^{k^\dagger} H = \lambda_k v^{k^\dagger}$$

Multiplying the first equation by $v^{k^\dagger}$ from the left and the second equation by v^i from the right we get:

$$\begin{aligned} v^{k^\dagger} H v^i &= \lambda_i v^{k^\dagger} v^i \\ v^{k^\dagger} H v^i &= \lambda_k v^{k^\dagger} v^i \end{aligned}$$

Subtracting, we get:

$$(\lambda_i - \lambda_k) v^{k^\dagger} v^i = 0$$

Thus either $\lambda_i = \lambda_j$ or the two eigenvectors are orthogonal ($v^{k^\dagger} v^k$ is the dot product of these two vectors).
For a $n \times n$ matrix, by collecting the n **normalized** eigenvectors another matrix may be formed:

$$V = (v^1 v^2 \cdots v^n)$$

In this matrix, $V_{ik} = (v^k)_i$, which means the i^{th} component of the k^{th} vector. This is the same as arranging the eigenvectors as columns in the matrix. We can also form a diagonal matrix of the eigenvalues:

$$\Lambda_{ik} = \Lambda_{ki} = \lambda_k \delta_{ik}$$

We can write an example of the eigenvalue equation (2.1) as:

$$\begin{array}{lcccccccccc} & \sum_j \alpha_{ij}(v^k)_j & = & \sum_j \alpha_{ij} V_{jk} & = & \lambda_k (v^k)_i & = & \lambda_k V_{ik} & = & \sum_j \lambda_k \delta_{jk} V_{ij} & = & \sum_j V_{ij} \Lambda_{jk} \\ \Rightarrow & & & & & \alpha V & = & V\Lambda & & & & \\ \Rightarrow & & & & & V^{-1}\alpha V & = & \Lambda & & & & \end{array}$$

Since the eigenvectors are orthogonal (for Hermitian matrices) and normalized when producing the transformation matrix V, $VV^\dagger = 1$. This is because each of the diagonal elements is really a dot product of one of the eigenvectors with itself. This again implies that the adjoint is the inverse and V is unitary. Thus the unitary transformation matrix V provides a similarity transformation which diagonalizes α, leaving the eigenvalues on the diagonal.

2.3 Reducible and irreducible representations

Let $\Gamma^{(1)}(A)$ and $\Gamma^{(2)}(A)$ be the matrices corresponding to two representations of the element A of the same group. The two representations may have different dimensions. Let us form the block diagonal matrix:

$$\Gamma(A) = \begin{pmatrix} \Gamma^{(1)}(A) & 0 \\ 0 & \Gamma^{(2)}(A) \end{pmatrix}$$

where the zeros represent rectangular null matrices with the correct numbers of rows and columns. Now upon matrix multiplication of two such matrices, the $\Gamma^{(1)}s$ will operate only among themselves and similarly for the $\Gamma^{(2)}s$. Thus the multiplication table for the new Γs will be the same and the new set of matrices is also a representation. Such a representation is called **reducible**. With the help of similarity transformations (which leave the multiplication table intact) it is now possible to mix up the rows and columns so thoroughly that the block diagonal form will not be apparent and thus finding whether some representation is reducible or not is a matter of some importance. If such blocks cannot be formed then the representation is **irreducible** and cannot be expressed with matrices of lower dimension. Reducible representations are represented as:

$$\Gamma = \Gamma^{(1)} + \Gamma^{(2)} + \cdots$$

Here the summation does not refer to matrix addition but is just a symbol to imply the block diagonalisable form. More generally, $\Gamma = \sum_i a_i \Gamma^{(i)}$ where a_i says how often $\Gamma^{(i)}$ appears in Γ. In quantum mechanics each irreducible representation will display the transformation properties of a set of degenerate eigenfunctions. The number of irreducible representations may give the number of distinct energy levels.

2.3.1 Orthogonality theorem

Lemma:

Any representation by matrices with non-vanishing determinants is equivalent (through a similarity transformation) to a representation by unitary matrices. The proof is through an explicit prescription for the construction of such a representation.
Proof: Let the matrix representing the i^{th} element be denoted by A_i. Now, even if the $A_i s$ are not Hermitian, $A_i A_i^\dagger$ is Hermitian because $(A_i A_i^\dagger)^\dagger = (A_i^\dagger)^\dagger A_i^\dagger = A_i A_i^\dagger$. From such individual Hermitian terms construct the Hermitian matrix $H = \sum_{i=1}^{i=h} A_i A_i^\dagger$, where h is the order of the group. Since all Hermitian matrices may be diagonalised, diagonalise H:

$$\begin{aligned} d &= U^{-1}HU = \sum_i U^{-1}A_i A_i^\dagger U \\ &= \sum_i U^{-1}A_i \underbrace{UU^{-1}}_{=E} A_i^\dagger U \\ &= \sum_i A_i' A_i'^\dagger \end{aligned}$$

Now multiplication by the adjoint means that each element is multiplied by the complex conjugate of itself. Thus the diagonal matrix d has only *real* and *positive* elements. Now:

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = \begin{pmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix}$$

Thus diagonal matrices multiply like ordinary numbers. Since d has real and positive elements, it is possible to define the matrices $\sqrt{d}$ and $\frac{1}{\sqrt{d}}$ just by taking the appropriate powers of the individual elements. Again, diagonal matrices commute. Therefore:

$$\begin{aligned} E &= d^{-1}d \\ &= d^{-\frac{1}{2}}dd^{-\frac{1}{2}} \\ &= d^{-\frac{1}{2}}\sum_i A'_i A'^{\dagger}_i d^{-\frac{1}{2}} \end{aligned} \tag{2.2}$$

Now construct the doubly transformed matrices:

$$A''_j = d^{-\frac{1}{2}}A'_j d^{\frac{1}{2}}$$

Thus:

$$A''_j A''^{\dagger}_j = d^{-\frac{1}{2}}A'_j d^{\frac{1}{2}}E \qquad \underbrace{d^{\frac{1}{2}}A^{\dagger}_j d^{-\frac{1}{2}}}_{order\ reversed \to adjoint}$$

Now expand the E in the middle:

$$\begin{aligned} A''_j A''^{\dagger}_j &= d^{-\frac{1}{2}}A'_j \underline{d^{\frac{1}{2}}[d^{-\frac{1}{2}}}\sum_i A'_i A'^{\dagger}_i \underline{d^{-\frac{1}{2}}]d^{\frac{1}{2}}}A'^{\dagger}_j d^{-\frac{1}{2}} \\ &= d^{-\frac{1}{2}}\sum_i A'_j A'_i A'^{\dagger}_i A'^{\dagger}_j d^{-\frac{1}{2}} \\ &= d^{-\frac{1}{2}}\sum_i A'_j A'_i (A'_j A'_i)^{\dagger} d^{-\frac{1}{2}} \end{aligned}$$

But $A'_j A'_i$ is just another element A'_k of the group (by the rearrangement theorem). Then remembering that the summation variable changes because the same j acts on all the i-s (and so there is really only one index that varies, i), and using (2.2)

$$A''_j A''^{\dagger} = d^{-\frac{1}{2}}\sum_k A'_k A'^{\dagger}_k d^{-\frac{1}{2}} = E$$

Thus A''_j is unitary.
Thus the prescription for constructing a unitary representation from any other representation is:

1. Form the sum $H = \sum_i A_i A^{\dagger}_i$.
2. Diagonalise it and find the diagonalising transformation U.
3. Transform the matrix elements by $U^{-1}A_i U$ to get the A'_is.
4. Take the root of the diagonalised matrix to get $d^{\frac{1}{2}}$ and $d^{-\frac{1}{2}}$.
5. Transform A'_i to A''_i by $A''_i = d^{-\frac{1}{2}}A'_i d^{\frac{1}{2}}$. These A''_is will be unitary and still form a representation of the group.

2.3.1.1 Schur's Lemma

Any matrix which commutes with *all* matrices of an *irreducible* representation *must* be constant matrix (i.e. of the form cE). Thus, if a non-constant commuting matrix exists, then the representation is reducible.

Proof:

We first transform any representation to consist only of unitary matrices. Let a matrix M commute with all the matrices in this representation. Thus

$$A_iM = MA_i \quad i = 1, 2, \cdots, h$$

Take the adjoint of both sides

$$M^\dagger A_i^\dagger = A_i^\dagger M^\dagger$$

Pre- and post- multiply by A_i

$$A_iM^\dagger \underbrace{A_i^\dagger A_i}_{=E\,(unitary)} = \underbrace{A_iA_i^\dagger}_{=E\,(unitary)} M^\dagger A_i \Rightarrow A_iM^\dagger = M^\dagger A_i$$

Thus, if M commutes, $M^\dagger$ also commutes.

Two Hermitian matrices can be formed by $H_1 = M + M^\dagger$ and $H_2 = i(M - M^\dagger)$. Since they contain only M and $M^\dagger$, they also commute with all A_i. Now $M = H_1 - iH_2$. To show that M is a constant, we only need to show that a commuting Hermitian matrix H is constant.

Diagonalise H (15 - all Hermitian matrices are diagonalisable)by the unitary transformation U.

$$d = U^{-1}HU$$

Let $A_i' = U^{-1}HU$, i.e. A_is are transformed to A_i's by the same transformation. Since matrix equations remain invariant under similarity transformations, the commutation relations between H and the A_is still hold:

$$A_i'd = dA_i'$$

Consider the $\mu\nu^{th}$ element:

$$\begin{aligned} (A_i')_{\mu\nu}d_{\nu\nu} &= d_{\mu\mu}(A_i')_{\mu\nu} \\ \Rightarrow (A_i')_{\mu\nu}(d_{\nu\nu} - d_{\mu\mu}) &= 0 \end{aligned} \tag{2.3}$$

We try to see what this implies through an explicit example. Let $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ and $d = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix}$. Then $A \times d = \begin{pmatrix} ax & by & cz \\ dx & ey & fz \\ gx & hy & iz \end{pmatrix}$ and

$d \times A = \begin{pmatrix} ax & bx & cx \\ dy & ey & fy \\ gz & hz & iz \end{pmatrix}$. These two may be equal only if $x = y = z$, which means that the matrix d is constant. If $x \neq y \neq z$, the matrix A must be diagonal, i.e., reducible. Even if some of the elements of d are different, say $x = y \neq z$, then also A must at least be of the form $\begin{pmatrix} a & b & 0 \\ d & e & 0 \\ 0 & 0 & i \end{pmatrix}$, which is also block diagonal, and hence, reducible.

Theorem:

If we consider *two* irreducible representations of the same group $\Gamma^{(1)}(A_i)$ and $\Gamma^{(2)}(A_i)$ of dimensionality l_1 and l_2 and if a rectangular matrix M exists such that $M\Gamma^{(1)}(A_i) = \Gamma^{2)}(A_i)M$ where $i = 1, 2, \cdots, h$, then:

1. If $l_1 \neq l_2$, $M = 0$.
2. If $l_1 = l_2$, then either $M = 0$ or $|M| \neq 0$.

Proof:

Consider only unitary representations (this can always be done, as shown above). Also, let $l_1 \leq l_2$ (depends on whom you call l_1). Take adjoints:

$$\begin{aligned} \Gamma^{(1)}(A_i)^\dagger M^\dagger &= M^\dagger \Gamma^{(2)}(A_i)^\dagger \\ \Rightarrow \underbrace{\Gamma^{(1)}(A_i)^{-1}}_{Unitary} M^\dagger &= M^\dagger \Gamma^{(2)}(A_i)^{-1} \end{aligned}$$

Again, due to isomorphism, $\Gamma^{(1)}(A_i)^\dagger = \Gamma^{(1)}(A_i^{-1})$ (inverse of a matrix in a particular representation is the matrix corresponding to the inverse element). Thus:

$$\Gamma^{(1)}(A_i^{-1})M^\dagger = M^\dagger \Gamma^{(2)}(A_i^{-1})$$

Premultiplying both sides by M:

$$\begin{aligned} \underbrace{M\Gamma^{(1)}(A_i^{-1})}_{\Gamma^{(2)}(A_i^{-1})M\, by\, basic\, premise} \quad M^\dagger &= MM^\dagger \Gamma^{(2)}(A_i^{-1}) \\ \Rightarrow \Gamma^{(2)}(A_i^{-1})MM^\dagger &= MM^\dagger \Gamma^{(2)}(A_i^{-1}) \end{aligned}$$

Thus $MM^\dagger$ commutes with *all* matrices of the representation. By Schur's lemma, it has to be a constant matrix, i.e. $MM^\dagger = cE$.

Let $l_1 = l_2$, i.e. the matrix M is square. Then $det\, MM^\dagger = |cE| = c^{l_1}$. Now if $c \neq 0$, $|MM^\dagger| \neq 0 \Rightarrow |M||M^\dagger| \neq 0$ i.e. $|M| \neq 0$ and M has an inverse. Taking original premise and pre-multiplying by $M^\dagger$:

$$\begin{aligned} M^{-1}M\Gamma^{(1)}A_i &= M^{-1}\Gamma^{(2)}(A_i)M \\ \Rightarrow \Gamma^{(1)}(A_i) &= M^{-1}\Gamma^{(2)}(A_i)M \end{aligned}$$

So the $\Gamma^{(1)}$ & $\Gamma^{(2)}$ are related by similarity transformations and the two representations are equivalent.

If $c = 0$, $MM^\dagger = \mathbf{0} \Rightarrow \sum_\lambda M_{\mu\lambda}M^\dagger_{\lambda\nu} = \sum_\lambda M_{\mu\lambda} \underbrace{M^*_{\nu\lambda}}_{adjoint} = 0$ for all μ, ν.

Consider $\mu = \nu$. Then $\sum_\lambda |M_{\mu\lambda}|^2 = 0$. This is possible only if *all* $M_{\mu\lambda} = 0$(sum of positive terms). Therefore, $M = \mathbf{0}$.

When $l_1 < l_2$, M has $l_1$columns and l_2 rows. We fill M out to a square $l_2 \times l_2$ matrix N by inserting $(l_2 - l_1)$ columns of zeros. Inspection shows that $NN^\dagger = MM^\dagger$[1]. But N has zero determinant because one column contains zeros only. Thus $NN^\dagger$ and hence $MM^\dagger$ also have zero determinant (determinant of product is equal to the product of determinants). But $MM^\dagger = cE$, so c has to be zero for the determinant to vanish. Then, using the same logic as above, $M = \mathbf{0}$.

2.3.1.2 Grand Orthogonality Theorem

If we consider *all* the *irreducible*, *inequivalent*, *unitary* representations of a group, then:

$$\sum_R \underbrace{\Gamma^{(i)}(R)^*_{\mu\nu}}_{\substack{Complex\,conjugate\,of\,the \\ \mu\nu^{th}\,element\,of\,the\,matrix \\ corresponding\,to\,the\,R^{th} \\ group\,element\,and\,belonging \\ to\,the\,i^{th}\,representation}} \Gamma^{(j)}(R)_{\alpha\beta} = \frac{h}{l_i}\delta_{ij}\delta_{\mu\alpha}\delta_{\nu\beta} \tag{2.4}$$

Here l_i is the dimensionality of $\Gamma^{(i)}$ and the summation is over all the elements of the group.

[1]For instance, let $M = \begin{pmatrix} a & b \\ c & d \\ x & y \end{pmatrix}$. Then $M^\dagger = \begin{pmatrix} a^* & c^* & x^* \\ b^* & d^* & y^* \end{pmatrix}$, and $N = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ x & y & 0 \end{pmatrix}$. The equality $NN^\dagger = MM^\dagger$ can be verified by direct matrix multiplication.

***Proof*:**

Consider two inequivalent representations $\Gamma^{(1)}$and $\Gamma^{(2)}$. Construct a matrix:

$$M = \sum_R \Gamma^{(2)}(R) X \Gamma^{(1)}(R^{-1})$$

where X is an arbitrary matrix with $l_1$columns and l_2 rows. Now:

$$\begin{aligned}
\Gamma^{(2)}(S)M &= \textstyle\sum_R \Gamma^{(2)}(S)\Gamma^{(2)}(R)X\Gamma^{(1)}(R^{-1}) \\
\Gamma\, belonging\, to & \quad Summation\, is\, over\, R,\ so\, S \\
2^{nd}\, repn. & \quad can\, go\, inside\, the\, summation \\
corresponding & \\
to\, the\, S\, group & \\
element & \\
&= \textstyle\sum_R \Gamma^{(2)}(S)\Gamma^{(2)}(R)X\Gamma^{(1)}(R^{-1})\Gamma^{(1)}(E) \\
&= \textstyle\sum_R \Gamma^{(2)}(S)\Gamma^{(2)}(R)X\Gamma^{(1)}(R^{-1})\Gamma^{(1)}(S^{-1}S) \\
&= \textstyle\sum_R \Gamma^{(2)}(S)\Gamma^{(2)}(R)X\Gamma^{(1)}(R^{-1})\Gamma^{(1)}(S^{-1})\Gamma^{(1)}(S) \\
&= \textstyle\sum_R \Gamma^{(2)}(SR)X\Gamma^{(1)}(R^{-1}S^{-1})\Gamma^{(1)}(S) \\
&= \left[\textstyle\sum_R \Gamma^{(2)}(SR)X\Gamma^{(1)}(SR)^{-1}\right]\Gamma^{(1)}(S)
\end{aligned}$$

By the rearrangement theorem, the product SR will generate the whole group as R goes over the whole set of elements. [2]So the summation will give the same result as a summation over R only. Thus the term above becomes:

$$\underbrace{\sum_R \Gamma^{(2)}(R)X\Gamma^{(1)}(R^{-1})}_{=M\, by\, definition}\Gamma^{(1)}(S)$$

Thus $\Gamma(2)(S)M = M\Gamma^{(1)}(S)$, i.e. M is the kind of matrix considered in the theorem above.

Since the representations are inequivalent, the only possible option is for M to be $\mathbf{0}$. Each element of M must be 0. Take one typical term:

$$M_{\alpha\mu} = 0 = \sum_R \sum_{\kappa\lambda} \Gamma^{(2)}(R)_{\alpha\kappa} X_{\kappa\lambda} \Gamma^{(1)}(R^{-1})_{\lambda\mu}$$

---

[2]Since the same S is being multiplied with each R, and since each row and column contain each element once and once only, if R traverses the whole group, SR also traverses the same group, but in some other order.

But X is arbitrary. So we are free to choose any X at our convenience. We choose the matrix X such that all $X_{\kappa\lambda} = 0$ except one, $X_{\beta\nu}$, which we put equal to 1. So only the term in M acting on $X_{\beta\nu}$ survives. Thus:

$$\begin{aligned} \textstyle\sum_R \Gamma^{(2)}(R)_{\alpha\beta} \underbrace{X_{\beta\nu}}_{=1} \Gamma^{(1)}(R^{-1})_{\nu\mu} &= 0 \\ \Rightarrow \textstyle\sum_R \Gamma^{(2)}(R)_{\alpha\beta}\Gamma^{(1)}(R^{-1})_{\nu\mu} &= 0 \end{aligned} \tag{2.5}$$

The Γs are unitary matrices. Thus:

$$\begin{aligned} \Gamma^{(1)}(R^{-1})_{\nu\mu} &= \Gamma^{(1)}(R)^{-1}_{\nu\mu} \\ &= \Gamma^{(1)}(R)^{\dagger}_{\nu\mu} \\ &= \Gamma^{(1)}(R)^{*}_{\mu\nu} \end{aligned}$$

Thus (2.5) becomes, remembering that order of products does not matter because we are dealing with components:

$$\sum_R \Gamma^{(1)}(R)^{*}_{\mu\nu}\Gamma^{(2)}(R)_{\alpha\beta} = 0$$

Thus, if $i \neq j$ the product is zero, which accounts for the δ_{ij} term.

If $i = j$, considering the first representation, say,

$$M = \sum_R \Gamma^{(1)}(R)X\Gamma^{(1)}(R^{-1})$$

M commutes with all the matrices of the representation. By Schur's lemma, $M = cE$. Take the $\mu\mu'$ element:

$$\sum_{\kappa\lambda}\sum_R \Gamma^{(1)}(R)_{\mu\kappa}X_{\kappa\lambda}\Gamma^{(1)}(R^{-1})_{\lambda\mu'} = c\delta_{\mu\mu'}$$

Choose all $X_{\kappa\lambda} = 0$ except $X_{\nu\nu'}$which we set equal to 1. Only one term survives in the $\kappa\lambda$ summation:

$$\sum_R \Gamma^{(1)}(R)_{\mu\nu}\Gamma^{(1)}(R^{-1})_{\nu'\mu'} = c_{\nu\nu'}\delta_{\mu\mu'} \tag{2.6}$$

Here $c_{\nu\nu'}$ is not really subscripted in the normal sense, but ν and ν' are tags which mark the c which corresponds to the particular X that we have chosen.

Now choose $\mu' = \mu$ and sum over μ:

$$\sum_R\sum_\mu \Gamma^{(1)}(R)_{\mu\nu}\Gamma^{(1)}(R^{-1})_{\nu'\mu} = c_{\nu\nu'}\sum_\mu \delta_{\mu\mu}$$

Now interchange the order of products (since these are only components):

$$\sum_R \sum_\mu \Gamma^{(1)}(R^{-1})_{\nu'\mu}\Gamma^{(1)}(R)_{\mu\nu} = c_{\nu\nu'} \sum_\mu \delta_{\mu\mu} = c_{\nu\nu'} l_1$$

because summation over the diagonal elements of the unit matrix gives l_1 which is the dimension of the 1^{th}representation. Summing over the repeated index, we get:

$$\sum_R \Gamma^{(1)}(\underbrace{R^{-1}R}_{=E})_{\nu'\nu} = l_1 c_{\nu'\nu}$$

Now, $\sum_R \Gamma^{(1)}(E) = hE$, i.e. just h unit matrices summed together. Then $\sum_R \Gamma^{(1)}(E)_{\nu'\nu} = h\delta_{\nu'\nu}$. Thus $c_{\nu\nu'} = h\frac{\delta_{\nu'\nu}}{l_1}$. Substituting in (2.6) and interchanging $\mu$and μ' in the right:

$$\sum_R \Gamma^{(1)}(R)_{\mu\nu}\Gamma^{(1)}(R^{-1})_{\nu'\mu'} = \frac{h}{l_1}\delta_{\mu'\mu}\delta_{\nu'\nu} \tag{2.7}$$

But the Γs are unitary, i.e. $\Gamma^{(1)}(R^{-1})_{\nu'\mu'} = \Gamma^{(1)}(R)^*_{\mu'\nu'}$. Then (2.7) becomes, on interchanging terms:

$$\sum_R \Gamma^{(1)}(R)^*_{\mu'\nu'}\Gamma^{(1)}(R)_{\mu\nu} = \frac{h}{l_1}\delta_{\mu'\mu}\delta_{\nu'\nu}$$

Combining the above two parts, the *Great Orthjogonality Theorem* is proved.

2.3.1.3 Geometrical interpretation of the great orthogonality theorem

Let us arrange the matrices corresponding to the representations in the following manner:

- The matrices corresponding to each element side-by-side horizontally.
- The different representations one below the other vertically.

Let us form h-dimensional vectors by choosing one element from each matrix in a row, keeping the component indices and representation constant. The orthogonality theorem says that these vectors are orthogonal. How many such vectors can be formed? Since each matrix can supply l^2 elements, the total number of such possible vectors is $\sum_i l_i^2$. But the vectors are h-dimensional and in h-dimensional space the maximum possible number of orthogonal vectors is h. Thus:

$$\sum_i l_i^2 \leq h$$

In fact, it will be shown later that they are actually equal. Thus, if we know the number of elements of the group, we can often simply guess the number of possible representations and the dimension of each. For instance, in our example group, $h = 6$ and the only possible solution is $6 = 1^2 + 1^2 + 2^2$. Thus there will be three representations with dimensions 1, 1 and 2.

2.3.2 The Character of a Representation

All matrix representations related to each other through unitary transformations are equivalent and hence have the same set of traces. Now, the ***character*** of the j^{th} representation is the set of h numbers $\chi^{(j)}(E)$, $\chi^{(j)}(A_2)$, $\cdots$, $\chi^{(j)}(A_h)$, where

$$\chi^{(j)}(R) = Tr\Gamma^{(j)}(R) = \sum_{\mu=1}^{l_j} \Gamma^{(j)}(R)_{\mu\mu}$$

Here l_j is the dimensionality of the j^{th} representation. Thus, the set of traces of the matrices constituting a representation is the character of that representation.

In a class, all the elements are conjugate to each other. By the definition of conjugate elements, they are related to each other through similarity transformations. So all elements in a particular class have the same trace, within one representation and within all the equivalent representations. Thus the character of a representation can be specified by just giving the traces of matrices corresponding to each *class*, instead of each element. For the k^{th} class, the trace within the j^{th} representation is denoted by $\chi^{(j)}(\zeta_k)$. Now, the great orthogonality theorem gives:

$$\sum_R \Gamma^{(i)}(R)^*_{\mu\nu}\Gamma^{(j)}(R)_{\alpha\beta} = \frac{h}{l_i}\delta_{ij}\delta_{\mu\alpha}\delta_{\nu\beta}$$

Set $\nu = \mu$ and $\beta = \alpha$:

$$\sum_R \Gamma^{(i)}(R)^*_{\mu\mu}\Gamma^{(j)}(R)_{\alpha\alpha} = \frac{h}{l_i}\delta_{ij}\delta_{\mu\alpha}\delta_{\mu\alpha} = \frac{h}{l_i}\delta_{ij}\delta_{\mu\alpha}$$

Sum over μ and α $\cdots$ (remembering that the complex conjugate of a sum is equal to the sum of complex conjugates):

$$\sum_R \sum_{\mu=1}^{l_i} \Gamma^{(i)}(R)^*_{\mu\mu} \sum_{\alpha=1}^{l_j} \Gamma^{(j)}(R)_{\alpha\alpha} = \sum_R \chi^{(i)}(R)^*\chi^{(j)}(R) = \frac{h}{l_i}\delta_{ij} \underbrace{\sum_{\mu\alpha}\delta_{\mu\alpha}}_{=l_i\, for\, i=j} = h\delta_{ij} \tag{2.8}$$

Thus the characters form a set of orthogonal vectors in group-element space. Now we collect the group elements according to classes within which $\chi^{(j)}(R)$ are the same. Then, using N_k to represent the number of elements in the class ζ_k:

$$\sum_k \chi^{(i)}(\zeta_k)^* \chi^{(j)}(\zeta_k) N_k = h\delta_{ij}$$

Here the summation is over classes (index k). Thus the characters of the various irreducible representations also form an orthogonal vector system. Here, the axes are labeled by the classes ζ_k. Now, the number of mutually orthogonal vectors in n-dimensional space is n. ***Thus the number of irreducible representations (i.e. the range of i, j in δ_{ij}) $\leq$ number of classes (i.e. k).*** In fact, they are equal. Coupling this equality with the previous one $\rightarrow \sum_i l_i^2 = h$, the number and dimensionality of the possible irreducible representations can be worked out from the numbers of group elements and classes, without explicitly working out the representations.

2.3.2.1 Character Tables

The traces corresponding to the various classes in the different representations are enumerated in tabular form using character tables. The rows are labeled by the different irreducible representations and the columns by classes. The character table takes the general form:

	$N_1\zeta_1$	$N_2\zeta_2$	$N_3\zeta_3$	$\cdots$
$\Gamma^{(1)}$				
$\Gamma^{(2)}$				
$\Gamma^{(3)}$				
$\vdots$				

.

Here, N_k is the number of elements in the k^{th} class. The entries are $\chi^{(j)}(\zeta_k)$, i.e. the trace of the k^{th} class belonging to the j^{th} representation.

The character table for our example group (fig. 1.1) is given by:

	ζ_1	$3\zeta_2$	$2\zeta_3$
$\Gamma^{(1)}$	1	1	1
$\Gamma^{(2)}$	1	-1	1
$\Gamma^{(3)}$	2	0	-1

.

It can be verified that the rows are orthogonal if the N_ks are used as weight factors.

2.3.2.2 Second Orthogonality relation for Characters